LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – MATHEMATICS

FIRST SEMESTER – NOVEMBER 2002

MT 1800 / M 721 ALGEBRA

06.11.2002 Max.: 100 Marks

1.00 – 4.00

Answer ALL questions.

01. (a) (i) If $O(G) = p^n$ where p is a prime number then prove that $O(Z(G)) > 1$. Hence deduce that any group of order p^2 is abelian (8)

(OR)

(ii) Prove that any group of order 48 is not simple (8)

(b) (i) If $p^\alpha \mid O(G)$, p is a prime number then prove that G has a subgroup H with $O(H) = p^\alpha$

(ii) Prove that any two p–Sylow subgroups of a finite group G are conjugate (17)

(OR)

(iii) Prove that every finite abelian group is a direct product of cyclic groups. (17)

02. (a) (i) If R is the field of Real numbers prove that $R[x]/(x^2+1)$ is a field isomorphic to the field of complex numbers.

(ii) If f(x) and g(x) are primitive polynomials, prove that f(x). g(x) is also a primitive polynomial. (8)

(iii) Prove that M is a free A– module if and only if it is isomorphic to a quotient module of A^n for some positive integer n.

(iv) State and prove Nakayama's lemma for modules (8)

(b) (i) If a primitive polynomial f(x) can be factored as the product of two polynomials having rational coefficients, then prove that it can be factored as the product of two polynomials having integer coefficients.

(ii) State and prove Eisenstein criterion for irreducibility of polynomials over the rationals. Use it to prove the polynomial $1 + x + x^2 + x^3 + x^4$ is irreducible over the rationals. (17)

(OR)

(iii) Let R be a unique factorisation domain. Prove that a primitive polynomial f(x) in R[x] is irreducible as an element of R[x] if and only if f(x) is irreducible as an element of F[x] where F is the field of quotients of R.

(iv) When R is a unique factorization domain. Prove that a primitive polynomial p(x) in R[x] can be factored in a unique way as a product of irreducible elements in R[x]. (17)

03. (a) (i) If p(x) is irreducible in F[x] and if v is a root of p(x), then prove that F(v) is isomorphic to F'(w) where w is a root of P'(t). Moreover, prove that this isomorphism σ can so be chosen that $v\sigma = w$ and $\alpha\sigma = \alpha'$ for very $\alpha \in F$ (8)

(OR)

(ii) Define simple extension of a field. If F is a field of characteristic O and if a,b are algebric over F, then prove that there is an element $c \in F(a,b)$ such that $F(a,b) = F(c)$ (8)

(b) (i) Show that the element $a \in K$ is algebraic over F if and only if F(a) is a finite extension of F. (12)

(ii) State and prove Remainder Theorem (5)

(OR)

(iii) State and prove Wedderburn's Theorem on finite division rings. (17)

04. (a) (i) When do you call a group solvable ? (2)

(ii) If $p(x) \in F(x)$ is solvable by radicals over F, then prove that the Galois group over F of p(x) is a solvable group. (6)

(OR)

(iii) Prove that G(K,F) is a subgroup of the group of all automorphisms of K. (3)

(iv) If K is a finite extension of a field F then prove that G(K,F) is a finite group and $O(G(K,F)) \leq [K: f]$ (5)

b. (i) Let F be a field and let $F(x_1, \ldots x_n)$ be the field of rational functions in $x_1, x_2 \ldots x_n$ over F. Suppose that S is the field of symmetric rational function: then prove that

1) $[F(x_1, x_2 \ldots, x_n,) : S] = n!$
2) $G(F(x_1, x_2 \ldots, x_n), S) = S_n$, the symmetric group of degree n.
3) If $a_1, a_2 \ldots, a_n$, are the elementary symmetric functions in $x_1, x_2 \ldots, x_n$ then prove that $S = F(a_1, a_2 \ldots a_n)$
4) $F(x_1, x_2 \ldots, x_n)$ is the splitting field over $F(a_1, a_2 \ldots a_n) = S$ of the polynomial $t^n - a_1 t^{n-1} + a_2 t^{n-2} \ldots (-1)^n a_n.$

(OR)

ii) State and prove the fundamental Theorem of Galois theory (17)

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