LOYOLA COLLEGE (AUTONOMOUS), CHENNAI–600 034.
M.Sc. DEGREE EXAMINATION – MATHEMATICS
FOURTH SEMESTER – APRIL 2003
MT 4950/ M 1055 NUMBER THEORY

23.04.2003
1.00 – 4.00 Max: 100 Mark

01. (a) (i) If (a,m)=1, prove that $a^{\phi(m)} \equiv 1 (\text{mod} m)$. Hence deduce Fermats theorem.

(OR)

(ii) of n ≥1 , prove that $\sum_{d|n} \phi(d) = n$ (8)

(b) (i) State and prove Wilson's theorem

(ii) Salve the congruence $x^2 + x + 7 \equiv 0$ (mod 189)

(OR)

(iii) Solve $x^2 + x + 7 \equiv 0 (\text{mod } 7^3)$

(iv) Reduce the congruence $4x^2 + 2x + 1 \equiv 0$ (mod5) to the form $x^2 \equiv a$ (mod p) hence find the solutions (17)

02. a) (i) Let P be an odd prime with (a, p) = 1. Consider he least non–negative residues module p of the integers a, 2a, 3a, …. $\left\{\left(\frac{p-1}{2}\right)\right\} a.$

If n denotes the number of these residues that exceed p/2, then prove that the Legendre symbol $\left(\frac{a}{p}\right) = (-1)^n$.

(OR)

(ii) Find the value of the Legendre symbol $\left(\frac{-6}{61}\right)$

(iii) Find the highest power of 7 that divides 1000! (8)

b) (i) If p is an odd prime and (a,2p) = 1 then prove that the

Legendre symbol $\left(\frac{a}{p}\right) = (-1)^t \; where \; t = \sum_{j=1}^{\frac{(p-1)}{2}} \left[\frac{r^a a}{p}\right].$

(ii) Define the Jacobi symbol $\left(\frac{P}{\phi}\right)$. Pr$ove\ that \left(-\frac{1}{\phi}\right) = (-1)^{\frac{(\phi-1)}{2}}$

(OR)

(iii) If f(n) is a multiplicative function and if $F(n) = \sum_{d/n} f(d),$ then prove that F(n) in multiplicative.

(iv) Define the Moebius function μ(n) and prove that inversion formula that

if $F(n) = \sum_{d|n} f(d)$ for every positive integer n

then F(n) $= \sum_{d|n} \mu(d)\ F\left(\frac{n}{d}\right)^{d|n}$. (17)

03. (a) (i) Find all the integral solutions of the equation ax + by =c if they exist, where a, b, c and integers.

(OR)

(ii) Show that there exists at least one positive solution of ax + by = c if g = (a, b) satisfies the condition g|c and gc >ab. (8)

(b) (i) Prove that all solutions of 3x +5y =1 can be written in the form x = 2 + 5t, y = –1 –3t.

(ii) Define a primitive solution of $x^2 + y^2 = z^2$. Prove that the positive primitive solutions of $x^2 + y^2 = z^2$ with y even are give by $x = r^2 = s^2$, y = 2rs, $z = r^2 + s^2$, where r and s are arbitrary integers of opposite parity with r > s >0 and (r,s) =1.

(OR)

(ii) Prove that every positive integer is a sum of four squares of integers. (17)

04. (a) (i) If P(n) is the partition function, with the usual notation prove that $p_m^{(n)} = p_{m-1}^{(n)} + P_n^{(n-m)}$ if n ≥ m>1.

(ii) using the graph of a partition, prove that the usual notation prove that the number of partitions of n into m summands is the same as the number of partitions of n having largest sum m and m.

(OR)

(iii) State Euler's formula and use it prove that Euler's identify for any positive integer n. (8)

(b) (i) If n ≥ 0 then prove that

$$q^e(n) - q^o(n) = \begin{cases} (-1)^j \ if\ n = (3j^2 \pm j)/2 \quad for\ some\ j = 0,1, \\ O \qquad otherwise \end{cases}$$

(OR)

(ii) of $\phi_{m}(x) = \prod_{n=1}^{m}(1-x^{n})$ $for\ o \leq k \leq 1, prove\ that \sum_{n=0}^{\infty} p_{m}(n)\, x^{n} conver$

(iii) Prove that for o ≤ x<1, the series $\sum_{n=0}^{\infty} p(n)x^{n} converges\, to\, \phi(x)^{-1}$. (17)

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